1 - [Carrol 1.7] Imagine we have a tensor $X^{\mu\nu}$ and a vector V^μ with components:

$$X^{\mu\nu} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}, \qquad V^\mu = \begin{pmatrix} -1, & 2, & 0, & -2 \end{pmatrix}$$

find:

$$(a) X^{\mu}{}_{\nu} \qquad (b) X_{\mu}{}^{\nu} \qquad (c) X^{(\mu\nu)} := \frac{1}{2}(X^{\mu\nu} + X^{\nu\mu}) \qquad (d) X_{[\mu\nu]} := (X_{\mu\nu} - X_{\nu\mu}) \qquad (e) X^{\lambda}_{\lambda}$$

$$(f) V^\mu V_\mu \qquad (g) V_\mu X^{\mu\nu}$$

2- [Schutz 3.2.6] Suppose A^μ and B_μ are vectors, show that $T^{\mu\nu} := A^\mu B^\nu$ is a rank $(2,0)$ tensor. Prove that its trace T^μ_μ is an invariant.

3- Suppose A is antisymmetric $(2,0)$ tensor, B is a symmetric $(2,0)$ tensor, C an arbitrary $(2,0)$ and D an arbitrary $(2,0)$ tensor. Prove:

$$(a) A^{\alpha\beta} B_{\alpha\beta} = 0 \qquad (b) A^{\alpha\beta} C_{\alpha\beta} = A^{\alpha\beta} C_{[\alpha\beta]} \qquad (c) B_{\alpha\beta} D^{\alpha\beta} = B_{\alpha\beta} D^{(\alpha\beta)}$$

4 - Prove the identity $\Lambda^{\mu'}{}_{\mu} \Lambda_{\mu'}{}^{\nu} = \delta_{\mu}{}^{\nu}$, then use it to prove the invariance of $A^\mu B_\mu$.

5 - Consider these equations:

$$\Lambda^{\alpha}{}_{\gamma} \eta_{\alpha\beta} \Lambda^{\beta}{}_{\delta} = \eta_{\gamma\delta}$$
$$\Lambda_{\mu}{}^{\delta} \eta^{\mu\nu} \Lambda_{\nu}{}^{\sigma} = \eta^{\delta\sigma}$$

Show by index manipulation that the second equation follows the first. (Hint: one way is to multiply the left hand sides and right hand sides separately and simplify)

6 - If $F^{\alpha\beta}$ is antisymmetric on its two indices, show that

$$F_{\mu}{}^{\alpha}{}_{,\nu} F^{\nu}{}_{\alpha} = -F_{\mu\alpha,\beta} F^{\alpha\beta}$$

7 - Show that the second rank tensor F which is antisymmetric in one coordinate frame $(F_{\mu\nu} = -F_{\nu\mu})$ is antisymmetric in all frames. Show that contravariant components are also antisymmetric $(F^{\mu\nu} = -F^{\nu\mu})$ and that symmetry is also coordinate invariant.

8 - Show that if A is a timelike vector and B is orthogonal to A, B is spacelike. Can you say all vectors orthogonal to a spacelike vector are timelike?