1- Recall the Lagrangian you used to derive geodesic equations. Show that $\mathcal{L}$ is constant along any geodesics. i.e. that

$$\frac{d}{d\tau}(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}) = 0$$

for $x^\mu(\tau)$ a solution of the geodesic equation. What is the symmetry that by Noether's theorem, gives rise to this constant of motion?

2 -The covariant derivatives of a covector field A_μ and a $(0,2)$ tensor $B_{\mu\nu}$ field are:

$$\begin{aligned} \nabla_\mu A_\nu &= \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda \\ \nabla_\lambda B_{\mu\nu} &= \partial_\lambda B_{\mu\nu} - \Gamma^\rho_{\lambda\mu} B_{\rho\nu} - \Gamma^\rho_{\lambda\nu} B_{\mu\rho} \end{aligned} \tag{1}$$

(a) Show that, even though $\partial_\mu A_\nu$ is not a tensor (which is why the Γ-term is required in $\nabla_\mu A_\nu$), the curl (or rotation) $\partial_\mu A_\nu - \partial_\nu A_\mu$ is a tensor, then show that the covariant curl of a covector is equal to its ordinary curl,

$$\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$$

This provides an alternative argument for the fact that $\partial_\mu A_\nu - \partial_\nu A_\mu$ is a tensor.
(b) Show that the covariant derivative of the metric is zero, $\nabla_\lambda g_{\mu\nu} = 0$.

3- Establish

$$\Gamma'^\mu_{\alpha\beta} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\sigma}{\partial x'^\alpha}\frac{\partial x^\rho}{\partial x'^\beta}\Gamma^\nu_{\sigma\rho} - \frac{\partial x^\nu}{\partial x'^\alpha}\frac{\partial x^\sigma}{\partial x'^\beta}\frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\sigma} \tag{2}$$

by assuming that the quantity defined by

$$\nabla_\mu A^\alpha = \partial_\mu A^\alpha + \Gamma^\alpha_{\nu\mu} A^\nu \tag{3}$$

has the tensor character indicated. Take the partial derivative of

$$\delta'^\alpha_\mu = \frac{\partial x'^\alpha}{\partial x'^\mu} = \frac{\partial x'^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial x'^\mu}$$

with respect to x'^ν to establish the alternative form:

$$\Gamma'^\mu_{\alpha\beta} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\sigma}{\partial x'^\alpha}\frac{\partial x^\rho}{\partial x'^\beta}\Gamma^\nu_{\sigma\rho} + \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial^2 x^\nu}{\partial x'^\alpha \partial x'^\beta} \tag{4}$$

2 and 4 are equivalent.

4- [d'inverno 6.5] assuming 3 and $\nabla_\alpha \phi = \partial_\alpha \phi$ apply the Leibniz rule to covariant derivative of $A_\mu A^\mu$, where A^μ is arbitrary to verify1.

5- [Carroll 3.5] Consider a 2-Sphere with coordinates (θ, ϕ) and metric:

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2$$

Show that the line of constant longitude ($\phi =$ constant) are geodesics, and that the only line of constant latitude ($\theta =$ constant) that is a geodesic is the equator ($\theta = \pi/2$).