1- [d'inverno 7.13] Express $(\nabla_b \nabla_c - \nabla_b \nabla_c)X_a$ in terms of Riemann tensor. Use this result to prove that any Killing vector satisfies

$$g^{bc}\nabla_b \nabla_a X_c - R_{ab}X^b = 0$$

2 - In the course, you defined the Riemann curvature tensor via commutator of covariant derivatives,

$$[\nabla_\alpha, \nabla_\beta]V^\gamma = R^\gamma_{\delta\alpha\beta}V^\delta \quad , \quad [\nabla_\alpha, \nabla_\beta]T^{\gamma\delta} = R^\gamma_{\epsilon\alpha\beta}T^{\epsilon\delta} + R^\delta_{\epsilon\alpha\beta}T^{\gamma\epsilon}$$

and you defined its contractions, the Ricci tensor $R_{\alpha\beta} = R^\gamma_{\alpha\gamma\beta}$ and the Ricci scalar $R = g^{\alpha\beta}R_{\alpha\beta}$. The Riemann curvature tensor has the symmetries

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} = -R_{\beta\alpha\gamma\delta} \quad , \quad R_{\alpha[\beta\gamma\delta]} = 0 \Leftrightarrow R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0$$

These symmetries also imply Ricci tensor is symmetric, $R_{\alpha\beta} = R_{\beta\alpha}$ [see section 8.3 of the lecture note]
(a) Like any linear operator, the covariant derivative ∇_α satisfies the Jacobi identity

$$[\nabla_\alpha, [\nabla_\beta, \nabla_\gamma]] + \text{cyclic permutations} = 0$$

Show that this implies the *Bianchi identity*

$$\nabla_\alpha R_{\mu\nu\beta\gamma} + \text{cyclic permutations in}(\alpha, \beta, \gamma) = 0$$

The equation you saw in class as the Bianchi identity, is called non-differential Bianchi identity.

(b) By double-contraction of the Bianchi identity, deduce the *Contracted Bianchi identity*

$$\nabla_\alpha(2R^\alpha{}_\gamma - \delta^\alpha{}_\gamma R) = 0$$

and show that this is equivalent to the statement that the *Einstein tensor* $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ has vanishing covariant divergence, $\nabla^\alpha G_{\alpha\beta} = 0$

(c) By double-contracting the expression for $[\nabla_\alpha, \nabla_\beta]T^{\gamma\delta}$, show that for any tensor $T^{\alpha\beta}$ one has

$$[\nabla_\alpha, \nabla_\beta]T^{\alpha\beta} = 0$$

3 - The weak static field produced by a non-relativistic mass density ρ is $g_{00} = -(1 + 2\phi)$ with the identification $T_{00} = \rho$ [see section 1.1 and 19.1 of the lecture note]. So the Newtonian field equation $\Delta\phi = 4\pi G_N \rho$ can also be written as

$$\Delta g_{00} = -8\pi G_N T_{00}$$

This suggests that the weak-field equations for a general energy-momentum tensor take the form

$$E_{\alpha\beta} = \quad \{\text{Some tensor generalizing}(-\Delta g_{00})\}_{\alpha\beta} \quad = 8\pi G_N T_{\alpha\beta}$$

What properties do you think $E_{\alpha\beta}$ has? Try to relate $E_{\alpha\beta}$ to well-known Einstein tensor.

4 - Using Einstein-Hilbert action and varying it with respect to metric, Einstein field equations can be achieved. This actions uses the most obvious Lagrangian, the Ricci scalar. Why was Ricci scalar the choice for the action? Why not using a Lagrangan $\mathcal{L}(g, \partial g)$? Why not $R_{\alpha\beta}$?