1 - (a) Consider a static observer (sitting at fixed values of $(r > 2m, \theta, \phi)$) in the Schwarzschild geometry

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Determine his worldline 4-velocity $u^\alpha = dx^\alpha/d\tau$ and the covariant (tensorial, specifically vectorial) acceleration $a^\alpha = \dot{u}^\alpha + \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma$ and calculate $g_{\alpha\beta}a^\alpha a^\beta$.

(b) Consider a freely (and radially) falling observer in the Schwarzschild geometry, initially at rest at radius $r(\tau = 0) \equiv R > 2m$. Show that the proper time it would (formally) take him to reach $r = 0$ is (up to factors of c) given by

$$\tau = \pi(\frac{R^3}{8m})^{1/2}$$

This can be interpreted as an estimate for the time of complete collapse of a star under its own gravitational attraction.

2- In part (a) you determined the acceleration of a static observer in Schwarzschild (SS) coordinates. The acceleration vector a^α appeared to be non-singular in these coordinates. This is misleading, however its norm (a scalar) turned out to be singular as $r \to 2m$ because the relevant component $g_{rr} = f(r)^{-1}$ is singular there.

Now we want to see what happens in simplest coordinates in which metric is non-singular at $r = 2m$, namely in Eddington-Finkelstein (EF) coordinates,

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2d\Omega^2$$

(a) Calculate $a^\alpha = \dot{u}^\alpha + \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma$ for a static observer in EF coordinates.

(b) Alternatively, determine a^α in EF coordinates simply from the result in SS coordinates by using the coordinate transformation.

(c) Determine the norm $g_{\alpha\beta}a^\alpha a^\beta$ in EF coordinates, and show that the result is equal to that obtained previously in SS coordinates.

3 - [d'inverno 16.3] What is the character of the coordinates of

(a) (t, ρ, z, ϕ) in

$$ds^2 = \rho^{-2m}dt^2 - \rho^{-2m}\left[\rho^{2m^2}(d\rho^2 + dz^2) + \rho^2 d\phi^2\right]$$

(b) (u, r, x, y)in

$$ds^2 = x^2du^2 - 2dudr + 4rx^{-1}dudx - r^2dx^2 - x^2dy^2$$